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Common Hermitian least squares solutions of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ subject to inequality restrictions☆

Fengxia Zhang *, Ying Li, Jianli Zhao

College of Mathematics Science, Liaocheng University, Liaocheng 252059, Shandong Province, PR China

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ABSTRACT

In this paper, we give some closed-form formulas for calculating maximal and minimal ranks and inertias of $P-X$ with respect to X, where $P \in C_H^n$ is given, X is common Hermitian least squares solutions to matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$. As application, we derive necessary and sufficient conditions for $X > P(\geq P, < P, \leq P)$ in the Löwner partial ordering. In addition, we give identifying conditions for the existence of definite common Hermitian least squares solutions to matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\mathrm{H}}^{n}$ stand for the sets of all $m \times n$ complex matrices and all $n \times n$ complex Hermitian matrices, respectively. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ represent the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively. I_m denotes the identity matrix of order m. We write $A > 0 (A \geq 0)$ if A is Hermitian positive (nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B (A \geq B)$ in the Löwner partial ordering if $A - B$ is positive (nonnegative) definite. The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations

$$\text{(i) } AXA = A, \qquad \text{(ii) } XAX = X, \qquad \text{(iii) } (AX)^* = AX, \qquad \text{(iv) } (XA)^* = XA.$$

Further, denote $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. The ranks of E_A and F_A are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. The inertia of $A \in \mathbb{C}_H^m$ is defined to be the triplet

$$I_n(A) = \{i_+(A),\ i_-(A),\ i_0(A)\},$$

where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the partial inertia of A. For a matrix $A \in C_H^m$, both $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = m - r(A)$ hold.

Consider a pair of linear matrix equations

$$A_1XA_1^* = B_1, \qquad A_2XA_2^* = B_2, \tag{1.1}$$

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* Corresponding author.
E-mail addresses: mathxia@yahoo.com, zhfxia_913@163.com (F. Zhang).

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where $X \in \mathbb{C}^{n\times n}$ is an unknown matrix, and $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}_H^{m_j}$, $j = 1, 2$ are given matrices. Common Hermitian least squares solutions of (1.1) is the common Hermitian solutions of their normal equations. The purpose of this paper is to consider common Hermitian least squares solutions to (1.1) subject to inequality restrictions. In particular, we study the inequality

$$X > P\ (<P, \geq P, \leq P) \tag{1.2}$$

in the Löwner partial ordering by determining the maximal and minimal inertias of $P - X$, where $P \in \mathbb{C}_H^n$ is given.

We shall use the following results on ranks and inertias of matrices in the latter part of this paper.

Lemma 1.1 ([1]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then*

$$r(A, B) = r(A) + r(E_AB) = r(B) + r(E_BA),$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C),$$

$$r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r(E_BAF_C).$$

Some useful variations of Lemma 1.1 are given below:

$$r\begin{pmatrix} A & BF_P \\ E_QC & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{pmatrix} - r(P) - r(Q),$$

$$r\begin{pmatrix} E_PAF_Q & E_PB \\ CF_Q & D \end{pmatrix} = r\begin{pmatrix} A & B & P \\ C & D & 0 \\ Q & 0 & 0 \end{pmatrix} - r(P) - r(Q),$$

$$r\begin{pmatrix} M & N \\ E_PA & E_PB \end{pmatrix} = r\begin{pmatrix} M & N & 0 \\ A & B & P \end{pmatrix} - r(P),$$

$$r\begin{pmatrix} M & AF_P \\ N & BF_P \end{pmatrix} = r\begin{pmatrix} M & A \\ N & B \\ 0 & P \end{pmatrix} - r(P).$$

Lemma 1.2 ([2]). *Let S be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let H be a set consisting of Hermitian matrices over $\mathbb{C}_H^m$. Then,*

(a) *Under $m = n$, S has a nonsingular matrix if and only if $\max_{X\in S} r(X) = m$.*
(b) *Under $m = n$, all $X \in S$ are nonsingular if and only if $\min_{X\in S} r(X) = m$.*
(c) *$0 \in S$ if and only if $\min_{X\in S} r(X) = 0$.*
(d) *$S = \{0\}$ if and only if $\max_{X\in S} r(X) = 0$.*
(e) *All $X \in S$ have the same rank if and only if $\max_{X\in S} r(X) = \min_{X\in S} r(X)$.*
(f) *H has a matrix $X > 0 (X < 0)$ if and only if $\max_{X\in H} i_+(X) = m (\max_{X\in H} i_-(X) = m)$.*
(g) *All $X \in H$ satisfy $X > 0 (X < 0)$ if and only if $\min_{X\in H} i_+(X) = m (\min_{X\in H} i_-(X) = m)$.*
(h) *H has a matrix $X \geq 0 (X \leq 0)$ if and only if $\min_{X\in H} i_-(X) = 0 (\min_{X\in H} i_+(X) = 0)$.*
(i) *All $X \in H$ satisfy $X \geq 0 (X \leq 0)$ if and only if $\max_{X\in H} i_-(X) = 0 (\max_{X\in H} i_+(X) = 0)$.*
(j) *All $X \in H$ have the same positive index of inertia if and only if $\max_{X\in H} i_+(X) = \min_{X\in H} i_+(X)$.*
(k) *All $X \in H$ have the same negative index of inertia if and only if $\max_{X\in H} i_-(X) = \min_{X\in H} i_-(X)$.*

Lemma 1.3 ([2]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$ and $Q \in \mathbb{C}^{m\times n}$ be given, and $P \in \mathbb{C}^{m\times m}$ nonsingular. Then,*

$$i_\pm(PAP^*) = i_\pm(A),$$

$$i_\pm(\lambda A) = \begin{cases} i_\pm(A), & \text{if } \lambda > 0 \\ i_\mp(A), & \text{if } \lambda < 0, \end{cases}$$

$$i_\pm\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = i_\pm(A) + i_\pm(B),$$

$$i_\pm\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} = r(Q).$$

Lemma 1.4 ([2]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, and denote $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. Then*

$$i_{\pm}(M) = r(B) + i_{\pm}(E_B A E_B).$$

In particular,

(a) *If $A \geq 0$, then $i_+(M) = r(A, B)$ and $i_-(M) = r(B)$.*
(b) *If $A \leq 0$, then $i_+(M) = r(B)$ and $i_-(M) = r(A, B)$.*
(c) *$i_{\pm}(A) \leq i_{\pm}(M) \leq i_{\pm}(A) + r(B)$.*

Some useful formulas derived from Lemma 1.4 are given below

$$i_{\pm}\begin{pmatrix} A & BF_P \\ F_P B^* & 0 \end{pmatrix} = i_{\pm}\begin{pmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{pmatrix} - r(P),$$

$$i_{\pm}\begin{pmatrix} E_Q A E_Q & E_Q B \\ B^* E_Q & D \end{pmatrix} = i_{\pm}\begin{pmatrix} A & B & Q \\ B^* & D & 0 \\ Q^* & 0 & 0 \end{pmatrix} - r(Q).$$

The matrix equation $AXA^* = B$ and its solutions were extensively studied in the literature, see, e.g., in [3–10]. Concerning the consistency and general common Hermitian solutions of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, the following results are well-known, see, e.g., in [11,12].

Lemma 1.5 ([7]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_H^m$ be given. Then,*

(a) *There exists an $X \in \mathbb{C}_H^n$ such that $AXA^* = B$ if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, or equivalently, $AA^{\dagger}B = B$.*
(b) *Under $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, the general Hermitian solution of $AXA^* = B$ can be written as*

$$X = A^{\dagger}B(A^{\dagger})^* + F_A V + (F_A V)^*,$$

where $V \in \mathbb{C}^{n\times n}$ is arbitrary.

Lemma 1.6 ([11]). *The pair of matrix equations in (1.1) have common solutions $X \in \mathbb{C}_H^n$ if and only if*

$$\mathscr{R}(B_j) \subseteq \mathscr{R}(A_j), \qquad r\begin{pmatrix} B_1 & 0 & A_1 \\ 0 & -B_2 & A_2 \\ A_1^* & A_2^* & 0 \end{pmatrix} = 2r(A), \quad j = 1, 2,$$

where $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. In this case, the general common Hermitian solutions of (1.1) can be written in the following parametric form

$$X = X_0 + F_A U_1 + (F_A U_1)^* + F_{A_1} U_2 F_{A_2} + (F_{A_1} U_2 F_{A_2})^*,$$

where X_0 is a special solution of (1.1), and $U_1, U_2 \in \mathbb{C}^{n\times n}$ are arbitrary.

The following results are related to ranks and inertias of some Hermitian matrix expressions.

Lemma 1.7 ([11,13]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ be given. Then*

$$\max_{X\in\mathbb{C}^{n\times m}} r[A - BX - (BX)^*] = \min\{m, r(P)\},$$

$$\min_{X\in\mathbb{C}^{n\times m}} r[A - BX - (BX)^*] = r(P) - 2r(B),$$

$$\max_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm}(P),$$

$$\min_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm}(P) - r(B),$$

where $P = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$.

Lemma 1.8 ([14]). *Let $p(X) = A - CXD - (CXD)^*$, where $A \in \mathbb{C}_H^m$, $C \in \mathbb{C}^{m\times p}$, $D \in \mathbb{C}^{q\times m}$. Then,*

$$\max_{X\in\mathbb{C}^{p\times q}} r[P(X)] = \min\{r(A, C, D^*),\ r(M_1),\ r(M_2)\},$$

$$\min_{X\in\mathbb{C}^{p\times q}} r[P(X)] = 2r(A,\ C,\ D^*) + \max\{r(M_1) - 2r(N_1),\ r(M_2) - 2r(N_2),\ s_+ + t_-,\ s_- + t_+\},$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_{\pm}[P(X)] = \min\{i_{\pm}(M_1),\ \ i_{\pm}(M_2)\},$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_{\pm}[P(X)] = r(A,\ C,\ D^*) + \max\{s_{\pm},\ t_{\pm}\},$$

where

$$M_1=\begin{pmatrix}A & C\\ \mathbb{C}^* & 0\end{pmatrix},\qquad N_1=\begin{pmatrix}A & C & D^*\\ \mathbb{C}^* & 0 & 0\end{pmatrix},\qquad M_2=\begin{pmatrix}A & D^*\\ D & 0\end{pmatrix},\qquad N_2=\begin{pmatrix}A & C & D^*\\ D & 0 & 0\end{pmatrix}$$

and $s_{\pm}=i_{\pm}(M_1)-r(N_1)$, $t_{\pm}=i_{\pm}(M_2)-r(N_2)$.

2. Ranks and inertias of $P-X$ subject to common Hermitian least squares solutions of $A_1XA_1^*=B_1$ and $A_2XA_2^*=B_2$

It is well known that the least squares solutions of matrix equation is the solution of its normal equation. Therefore, the common Hermitian least squares solutions of (1.1) is the common Hermitian solutions of matrix equations

$$A_1^*A_1XA_1^*A_1=A_1^*B_1A_1,\qquad A_2^*A_2XA_2^*A_2=A_2^*B_2A_2. \tag{2.1}$$

By Lemma 1.6, we can achieve the necessary and sufficient conditions of (1.1) having common Hermitian least squares solutions and the parametric form for the general common Hermitian least squares solutions.

Lemma 2.1. *The pair of matrix equations of (1.1) have a common Hermitian least squares solutions if and only if*

$$\mathscr{R}(A_j^*B_jA_j)\subseteq\mathscr{R}(A_j^*A_j)\quad\text{and}\quad r\begin{pmatrix}A_1^*B_1A_1 & 0 & A_1^*A_1\\ 0 & -A_2^*B_2A_2 & A_2^*A_2\\ A_1^*A_1 & A_2^*A_2 & 0\end{pmatrix}=2r(A),\quad j=1,2, \tag{2.2}$$

where $A^*=(A_1^*A_1,\ A_2^*A_2)$. *In this case, the general common Hermitian least squares solutions of (1.1) can be written in the following parametric form*

$$X=X_0+F_AU_1+F_AU_1^*+F_{A_1}U_2F_{A_2}+(F_{A_1}U_2F_{A_2})^*, \tag{2.3}$$

where X_0 is a special common Hermitian solution of (2.1), and $U_1, U_2\in\mathbb{C}^{n\times n}$ are arbitrary.

Proof. Because common Hermitian least squares solutions to (1.1) is the common Hermitian solutions of matrix equation (2.1), then (2.2) follows from Lemma 1.6. In addition, the parametric form of general common Hermitian least squares solutions of (1.1) is achieved as follow

$$X=X_0+F_AU_1+(F_AU_1)^*+F_{A_1^*A_1}U_2F_{A_2^*A_2}+(F_{A_1^*A_1}U_2F_{A_2^*A_2})^*,$$

where X_0 is a special solution of matrix equation (2.1), and $U_1, U_2\in\mathbb{C}^{n\times n}$ are arbitrary. Notice that

$$F_{A_1^*A_1}=I-(A_1^*A_1)^\dagger(A_1^*A_1)=I-A_1^\dagger(A_1^\dagger)^*A_1^*A_1=I-A_1^\dagger A_1=F_{A_1}.$$

Therefore, (2.3) is established. □

For convenience of representation, the following notation for the collection of all common Hermitian least squares solutions of (1.1) is adopted

$$S=\{X\in\mathbb{C}_H^n\mid A_1^*A_1XA_1^*A_1=A_1^*B_1A_1,\ A_2^*A_2XA_2^*A_2=A_2^*B_2A_2\}. \tag{2.4}$$

We also need the following preliminary result.

Lemma 2.2. *Let*

$$P(X,Y)=A-BX-(BX)^*-CYD-(CYD)^*, \tag{2.5}$$

where $A\in\mathbb{C}_H^m$, $B\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{m\times p}$ and $D\in\mathbb{C}^{q\times m}$ are given, and $X\in\mathbb{C}^{n\times m}$, $Y\in\mathbb{C}^{p\times q}$ are variable matrices. Also let

$$M=\begin{pmatrix}A & B & C & D^*\\ B^* & 0 & 0 & 0\end{pmatrix},\qquad M_1=\begin{pmatrix}A & B & C\\ B^* & 0 & 0\\ \mathbb{C}^* & 0 & 0\end{pmatrix},\qquad M_2=\begin{pmatrix}A & B & D^*\\ B^* & 0 & 0\\ D & 0 & 0\end{pmatrix},$$

$$N_1=\begin{pmatrix}A & B & C & D^*\\ B^* & 0 & 0 & 0\\ \mathbb{C}^* & 0 & 0 & 0\end{pmatrix},\qquad N_2=\begin{pmatrix}A & B & C & D^*\\ B^* & 0 & 0 & 0\\ D & 0 & 0 & 0\end{pmatrix}.$$

Then,

$$\max_{X,Y} r[P(X,Y)]=\min\{m, r(M), r(M_1), r(M_2)\}, \tag{2.6}$$

$$\min_{X,Y} r[P(X,Y)]=2r(M)-2r(B)+\max\{s_++s_-, s_-+t_+, s_++t_-, t_++t_-\}, \tag{2.7}$$

$$\max_{X,Y} i_{\pm}[P(X,Y)] = \min\{i_{\pm}(M_1), i_{\pm}(M_2)\}, \tag{2.8}$$

$$\min_{X,Y} i_{\pm}[P(X,Y)] = r(M) - r(B) + \max\{s_{\pm}, t_{\pm}\}, \tag{2.9}$$

where $s_{\pm} = i_{\pm}(M_1) - r(N_1)$ *and* $t_{\pm} = i_{\pm}(M_2) - r(N_2)$.

Proof. Applying Lemma 1.7 to variable matrix X in (2.5) and simplifying, we obtain

$$\max_{X} r[P(X,Y)] = \min\left\{m, r\begin{pmatrix} A - CYD - (CYD)^* & B \\ B^* & 0 \end{pmatrix}\right\}, \tag{2.10}$$

$$\min_{X} r[P(X,Y)] = r\begin{pmatrix} A - CYD - (CYD)^* & B \\ B^* & 0 \end{pmatrix} - 2r(B), \tag{2.11}$$

$$\max_{X} i_{\pm}[P(X,Y)] = i_{\pm}\begin{pmatrix} A - CYD - (CYD)^* & B \\ B^* & 0 \end{pmatrix}, \tag{2.12}$$

$$\min_{X} i_{\pm}[P(X,Y)] = i_{\pm}\begin{pmatrix} A - CYD - (CYD)^* & B \\ B^* & 0 \end{pmatrix} - r(B). \tag{2.13}$$

Let

$$q(Y) = \begin{pmatrix} A - CYD - (CYD)^* & B \\ B^* & 0 \end{pmatrix}.$$

This implies that

$$q(Y) = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} - \begin{pmatrix} C \\ 0 \end{pmatrix} Y(D, 0) - \left[\begin{pmatrix} C \\ 0 \end{pmatrix} Y(D, 0)\right]^*. \tag{2.14}$$

Applying Lemma 1.8 to (2.14) gives

$$\max_{Y} r[q(Y)] = \min\{r(M), r(M_1), r(M_2)\}, \tag{2.15}$$

$$\min_{Y} r[q(Y)] = 2r(M) + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\}, \tag{2.16}$$

$$\max_{Y} i_{\pm}[q(Y)] = \min\{i_{\pm}(M_1), i_{\pm}(M_2)\}, \tag{2.17}$$

$$\min_{Y} i_{\pm}[q(Y)] = r(M) + \max\{s_{\pm}, t_{\pm}\}, \tag{2.18}$$

where $s_{\pm} = i_{\pm}(M_1) - r(N_1)$, $t_{\pm} = i_{\pm}(M_2) - r(N_2)$. Substituting (2.15)–(2.18) into (2.10)–(2.13) yields (2.6)–(2.9). □

Theorem 2.3. *Let* $A_j \in \mathbb{C}^{m_j \times n}$, $B_j \in \mathbb{C}_H^{m_j}$, $j = 1, 2$ *and* $P \in \mathbb{C}_H^n$ *be given, and assume that* (1.1) *have a common Hermitian least squares solutions and* S *is as given in* (2.4). *Also, let*

$$T_1 = \begin{pmatrix} A_1^*A_1 & A_1^*A_1PA_1^*A_1 - A_1^*B_1A_1 & 0 \\ A_2^*A_2 & 0 & A_2^*B_2A_2 - A_2^*A_2PA_2^*A_2 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} A_1^*A_1 & A_1^*B_1A_1 - A_1^*APA_1^*A_1 \\ A_2 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} A_1 & 0 \\ A_2^*A_2 & A_2^*B_2A_2 - A_2^*A_2PA_2^*A_2 \end{pmatrix},$$

$$T_4 = A_1^*B_1A_1 - A_1^*A_1PA_1^*A_1, \qquad T_5 = A_2^*B_2A_2 - A_2^*A_2PA_2^*A_2.$$

Then,

$$\max_{X \in S} r(P - X) = \min\{n, c_1, c_2, c_3\}, \tag{2.19}$$

$$\min_{X \in S} r(P - X) = 2r(T_1) + \max\{s_1, s_2, s_3, s_4\}, \tag{2.20}$$

$$\max_{X \in S} i_{\pm}(P - X) = \min\{n + i_{\mp}(T_4) - r(A_1), n + i_{\mp}(T_5) - r(A_2)\}, \tag{2.21}$$

$$\min_{X \in S} i_{\pm}(P - X) = r(T_1) + \max\{i_{\mp}(T_4) - r(T_2), i_{\mp}(T_5) - r(T_3)\}, \tag{2.22}$$

where

$$c_1 = 2n + r(T_1) - r(A) - r(A_1) - r(A_2), \qquad c_2 = 2n + r(T_4) - 2r(A_1),$$

$$c_3 = 2n + r(T_5) - 2r(A_2), \qquad s_1 = r(T_4) - 2r(T_2), \qquad s_2 = r(T_5) - 2r(T_3),$$

$$s_3 = i_-(T_4) + i_+(T_5) - r(T_2) - r(T_3), \qquad s_4 = i_+(T_4) + i_-(T_5) - r(T_2) - r(T_3).$$

Proof. From (2.3), the difference $P - X$ can be expressed as

$$P - X = P - X_0 - F_AU_1 - (F_AU_1)^* - F_{A_1}U_2F_{A_2} - (F_{A_1}U_2F_{A_2})^*. \tag{2.23}$$

Let

$$L = \begin{pmatrix} P - X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} P - X_0 & F_A & F_{A_1} \\ F_A & 0 & 0 \\ F_{A_1} & 0 & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} P - X_0 & F_A & F_{A_2} \\ F_A & 0 & 0 \\ F_{A_2} & 0 & 0 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} P - X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} P - X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 \end{pmatrix}.$$

Applying Lemma 2.2 to (2.23) yields

$$\max_{X\in S} r(P - X) = \min\{n, r(L), r(G_1), r(G_2)\}, \tag{2.24}$$

$$\min_{X\in S} r(P - X) = 2r(L) + \max\{t_1, t_2, t_3, t_4\} - 2r(F_A), \tag{2.25}$$

$$\max_{X\in S} i_\pm(P - X) = \min\{i_\pm(G_1), i_\pm(G_2)\}, \tag{2.26}$$

$$\min_{X\in S} i_\pm(P - X) = r(L) + \max\{i_\pm(G_1) - r(L_1), i_\pm(G_2) - r(L_2)\} - r(F_A), \tag{2.27}$$

where

$$t_1 = r(G_1) - 2r(L_1), \qquad t_2 = r(G_2) - 2r(L_2), \tag{2.28}$$

$$t_3 = i_+(G_1) + i_-(G_2) - r(L_1) - r(L_2), \tag{2.29}$$

$$t_4 = i_-(G_1) + i_+(G_2) - r(L_1) - r(L_2). \tag{2.30}$$

We will simplify $r(L), r(L_1), r(L_2), i_\pm(G_1), i_\pm(G_2)$ by the three types of elementary block matrix operations, elementary block congruence matrix operations and, Lemmas 1.1 and 1.4. It is easy to show that $\mathscr{R}(F_A) \subseteq \mathscr{R}(F_{A_1}), \mathscr{R}(F_A) \subseteq \mathscr{R}(F_{A_2})$. Therefore, we obtain

$$\begin{aligned}
r(L) &= r\begin{pmatrix} P - X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} P - X_0 & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} P - X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & A^* \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{pmatrix} - r(A) - r(A_1) - r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & 0 \\ A_2 & A_2(X_0 - P)A^* \end{pmatrix} - r(A) - r(A_1) - r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & 0 & 0 \\ A_2 & A_2(X_0 - P)A_1^*A_1 & A_2(X_0 - P)A_2^*A_2 \end{pmatrix} - r(A) - r(A_1) - r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & -A_1(X_0 - P)A_1^*A_1 & 0 \\ A_2 & 0 & A_2(X_0 - P)A_2^*A_2 \end{pmatrix} - r(A) - r(A_1) - r(A_2) \\
&= 2n + r(T_1) - r(A) - r(A_1) - r(A_2),
\end{aligned}$$

$$
\begin{aligned}
r(L_1) &= r\begin{pmatrix} P-X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} P-X_0 & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} P-X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & A_1^* \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{pmatrix} - 2r(A_1) - r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & -A_1(X_0-P)A_1^* \\ A_2 & 0 \end{pmatrix} - 2r(A_1) - r(A_2) \\
&= 2n + r\begin{pmatrix} A_1^*A_1 & A_1^*A_1(X_0-P)A_1^*A_1 \\ A_2 & 0 \end{pmatrix} - 2r(A_1) - r(A_2) = 2n + r(T_2) - 2r(A_1) - r(A_2),
\end{aligned}
$$

$$
\begin{aligned}
r(L_2) &= r\begin{pmatrix} P-X_0 & F_A & F_{A_1} & F_{A_2} \\ F_A & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} P-X_0 & F_{A_1} & F_{A_2} \\ F_{A_2} & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} P-X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & A_2^* \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{pmatrix} - r(A_1) - 2r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & 0 \\ A_2 & A_2(X_0-P)A_2^* \end{pmatrix} - r(A_1) - 2r(A_2) \\
&= 2n + r\begin{pmatrix} A_1 & 0 \\ A_2^*A_2 & A_2^*A_2(X_0-P)A_2^*A_2 \end{pmatrix} - r(A_1) - 2r(A_2) = 2n + r(T_3) - r(A_1) - 2r(A_2),
\end{aligned}
$$

$$
\begin{aligned}
i_\pm(G_1) &= i_\pm\begin{pmatrix} P-X_0 & F_A & F_{A_1} \\ F_A & 0 & 0 \\ F_{A_1} & 0 & 0 \end{pmatrix} = i_\pm\begin{pmatrix} P-X_0 & F_{A_1} \\ F_{A_1} & 0 \end{pmatrix} = i_\pm\begin{pmatrix} P-X_0 & I_n & 0 \\ I_n & 0 & A_1^* \\ 0 & A_1 & 0 \end{pmatrix} - r(A_1) \\
&= i_\pm\begin{pmatrix} 0 & I_n & -\frac{1}{2}(P-X_0)A_1^* \\ I_n & 0 & A_1^* \\ -\frac{1}{2}A_1(P-X_0) & A_1 & 0 \end{pmatrix} - r(A_1) \\
&= i_\pm\begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & A_1(P-X_0)A_1^* \end{pmatrix} - r(A_1) \\
&= n + i_\pm(A_1^*A_1(P-X_0)A_1^*A_1) - r(A_1) \\
&= n + i_\pm(A_1^*A_1PA_1^*A_1 - A_1^*B_1A_1) - r(A_1) = n + i_\mp(T_4) - r(A_1).
\end{aligned}
$$

Similarly, we obtain

$$
i_\pm(G_2) = i_\pm\begin{pmatrix} P-X_0 & F_A & F_{A_2} \\ F_A & 0 & 0 \\ F_{A_2} & 0 & 0 \end{pmatrix} = n + i_\pm(A_2^*A_2PA_2^*A_2 - A_2^*B_2A_2) - r(A_2) = n + i_\mp(T_5) - r(A_2).
$$

Therefore, we have

$$
r(G_1) = i_+(G_1) + i_-(G_1) = 2n + r(T_4) - 2r(A_1), \qquad r(G_2) = i_+(G_2) + i_-(G_2) = 2n + r(T_5) - 2r(A_2).
$$

Substituting the above results into (2.28)–(2.30), we obtain

$$t_1 = r(T_4) - 2r(T_2) + 2r(A_1) + 2r(A_2) - 2n, \tag{2.31}$$
$$t_2 = r(T_5) - 2r(T_3) + 2r(A_1) + 2r(A_2) - 2n, \tag{2.32}$$
$$t_3 = i_-(T_4) + i_+(T_5) + 2r(A_1) + 2r(A_2) - r(T_2) - r(T_3) - 2n, \tag{2.33}$$
$$t_4 = i_+(T_4) + i_-(T_5) + 2r(A_1) + 2r(A_2) - r(T_2) - r(T_3) - 2n. \tag{2.34}$$

Substituting (2.31)–(2.34) into (2.24)–(2.27) yields (2.19)–(2.22). □

If $P = 0$ in (2.19)–(2.22), then the maximal and minimal ranks and inertias of the common Hermitian least squares solutions to (1.1) are given as follows.

Corollary 2.4. *Let $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}_H^{m_j}$, $j = 1, 2$ be given, and assume that (1.1) have common Hermitian least squares solutions. Also, define*

$$
H_1 = \begin{pmatrix} A_1^*A_1 & A_1^*B_1A_1 & 0 \\ A_2^*A_2 & 0 & A_2^*B_2A_2 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} A_1^*A_1 & A_1^*B_1A_1 \\ A_2 & 0 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} A_1 & 0 \\ A_2^*A_2 & A_2^*B_2A_2 \end{pmatrix}.
$$

Then,

$$\max_{X\in S} r(X) = \min\{n,\ d_1, d_2, d_3\}$$

$$\min_{X\in S} r(X) = 2r(H_1) + \max\{v_1, v_2, v_3, v_4\},$$

$$\max_{X\in S} i_{\pm}(X) = \min\{n + i_{\pm}(A_1^*B_1A_1) - r(A_1), n + i_{\pm}(A_2^*B_2A_2) - r(A_2)\},$$

$$\min_{X\in S} i_{\pm}(X) = r(H_1) + \max\{i_{\pm}(A_1^*B_1A_1) - r(H_2), i_{\pm}(A_2^*B_2A_2) - r(H_3)\},$$

where

$$d_1 = 2n + r(H_1) - r(A) - r(A_1) - r(A_2),$$

$$d_2 = 2n + r(A_1^*B_1A_1) - 2r(A_1), d_3 = 2n + r(A_2^*B_2A_2) - 2r(A_2),$$

$$v_1 = r(A_1^*B_1A_1) - 2r(H_2), v_2 = r(A_2^*B_2A_2) - 2r(H_3),$$

$$v_3 = i_-(A_1^*B_1A_1) + i_+(A_2^*B_2A_2) - r(H_2) - r(H_3),$$

$$v_4 = i_+(A_1^*B_1A_1) + i_-(A_2^*B_2A_2) - r(H_2) - r(H_3).$$

3. Definite common Hermitian least squares solutions of $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$

Applying Lemma 1.2 to (2.19)–(2.22), we obtain necessary and sufficient conditions for the existence of the common Hermitian least squares solutions of (1.1) such that $X > P$ $(\geq P, < P, \ \leq P)$.

Theorem 3.1. *Assume that (1.1) have a common Hermitian least squares solution, and all signs are stated as in Theorem 2.3. Then*

(a) *Eq. (1.1) have a common Hermitian least squares solution $X > P$ if and only if*

$$i_+(T_4) = r(A_1), \qquad i_+(T_5) = r(A_2). \tag{3.1}$$

(b) *Eq. (1.1) have a common Hermitian least squares solution $X < P$ if and only if*

$$i_-(T_4) = r(A_1), \qquad i_-(T_5) = r(A_2). \tag{3.2}$$

(c) *Eq. (1.1) have a common Hermitian least squares solution $X \geq P$ if and only if*

$$r(T_1) = r(T_2) = r(T_3), \qquad T_4 \geq 0, \qquad T_5 \geq 0. \tag{3.3}$$

(d) *Eq. (1.1) have a common Hermitian least squares solution $X \leq P$ if and only if*

$$r(T_1) = r(T_2) = r(T_3), \qquad T_4 \leq 0, \qquad T_5 \leq 0. \tag{3.4}$$

(e) *There exist nonsingular matrix $P - X$ such that X is common Hermitian least squares solution to (1.1) if and only if*

$$n + r(T_1) \geq r(A) + r(A_1) + r(A_2), \qquad n + r(T_4) \geq 2r(A_1) \quad \text{and} \quad n + r(T_5) \geq 2r(A_2). \tag{3.5}$$

Proof. We will only prove (a), (c) and (e), since (b) and (d) are similar with (a) and (c), respectively. Applying (f) in Lemma 1.2, Eq. (1.1) have a common least squares solution $X > P$ if and only if

$$\max_{X\in S} i_+(X - P) = n. \tag{3.6}$$

By Lemma 1.3, (3.6) is equivalent with

$$\max_{X\in S} i_-(P - X) = n. \tag{3.7}$$

Substituting (2.21) into (3.7) yields

$$i_+(T_4) = r(A_1), \qquad i_+(T_5) \geq r(A_2) \quad \text{or} \quad i_+(T_5) = r(A_2), \qquad i_+(T_4) \geq r(A_1). \tag{3.8}$$

For $T_4 = A_1^*B_1A_1 - A_1^*A_1PA_1^*A_1$, $T_5 = A_2^*B_2A_2 - A_2^*A_2PA_2^*A_2$, we have

$$i_+(T_4) \leq r(T_4) \leq r(A_1), \qquad i_+(T_5) \leq r(T_5) \leq r(A_2).$$

Therefore, (3.8) is equivalent with

$$i_+(T_4) = r(A_1), \qquad i_+(T_5) = r(A_2).$$

Thus (3.1) is obtained.

Similarly, Applying (h) in Lemma 1.2, then Eq. (1.1) have a common least squares solution $X \geq P$ if and only if

$$\min_{X\in S} i_-(X-P)=0 \quad \text{i.e.} \quad \min_{X\in S} i_+(P-X)=0. \tag{3.9}$$

Substituting (2.22) into (3.9) yields

$$r(T_1)+i_-(T_4)=r(T_2), \qquad r(T_1)+i_-(T_5)\leq r(T_3) \tag{3.10}$$

or

$$r(T_1)+i_-(T_5)=r(T_3), \qquad r(T_1)+i_-(T_4)\leq r(T_2). \tag{3.11}$$

Note that $r(T_1)\geq r(T_2)$, $r(T_1)+i_-(T_4)=r(T_2)$, so we have

$$r(T_1)=r(T_2), \qquad i_-(T_4)=0.$$

Also note that $r(T_1)\geq r(T_3)$, $r(T_1)+i_-(T_5)\leq r(T_3)$, so we obtain

$$r(T_1)=r(T_3), \qquad i_-(T_5)=0$$

Therefore, (3.10) is equivalent with

$$r(T_1)=r(T_2)=r(T_3), \qquad i_-(T_4)=0,\ i_-(T_5)=0. \tag{3.12}$$

By the same method, (3.11) is also equivalent with (3.12). Then (3.3) holds.

By (a) in Lemma 1.2, there exist nonsingular matrix $P-X$ such that X is common Hermitian least squares solution to (1.1) if and only if

$$\max_{X\in S} r(P-X)=n. \tag{3.13}$$

Substituting (2.19) into (3.13) yields

$$\min\{n, c_1, c_2, c_3\}=n, \tag{3.14}$$

so (3.14) is equivalent with

$$c_1=2n+r(T_1)-r(A)-r(A_1)-r(A_2)\geq n,$$
$$c_2=2n+r(T_4)-2r(A_1)\geq n, \quad \textit{and} \quad c_3=2n+r(T_5)-2r(A_2)\geq n.$$

Therefore, (3.5) holds. □

Accordingly, we can achieve equivalent conditions for the existence of common Hermitian positive (negative, nonpositive, nonnegative) definite least squares solutions to (1.1).

Corollary 3.2. *Assume that (1.1) have a common Hermitian least squares solution, and all signs are stated as in Corollary 2.4. Then*

(a) *Eq. (1.1) have a common positive definite Hermitian least squares solution if and only if*

$$i_+(A_1^*B_1A_1)=r(A_1), \qquad i_+(A_2^*B_2A_2)=r(A_2).$$

(b) *Eq. (1.1) have a common negative definite Hermitian least squares solution if and only if*

$$i_-(A_1^*B_1A_1)=r(A_1), \qquad i_-(A_2^*B_2A_2)=r(A_2).$$

(c) *Eq. (1.1) have a common nonnegative definite Hermitian least squares solution if and only if*

$$r(H_1)=r(H_2)=r(H_3), \qquad A_1^*B_1A_1\geq 0, \qquad A_2^*B_2A_2\geq 0.$$

(d) *Eq. (1.1) have a common nonpositive definite Hermitian least squares solution if and only if*

$$r(H_1)=r(H_2)=r(H_3), \qquad A_1^*B_1A_1\leq 0, \qquad A_2^*B_2A_2\leq 0.$$

(e) *There exist nonsingular common Hermitian least squares solution to (1.1) if and only if*

$$n+r(H_1)\geq r(A)+r(A_1)+r(A_2),$$
$$n+r(A_1^*B_1A_1)\geq 2r(A_1) \quad \textit{and} \quad n+r(A_2^*B_2A_2)\geq 2r(A_2).$$

Proof. Let $P=0$ in Theorem 3.1, Corollary 3.2 is achieved. □

4. Conclusions

In the previous sections, we have presented some closed-form formulas for calculating maximal and minimal ranks and inertias of $P-X$ with respect to X, where $P \in \mathbb{C}_H^n$ is given, X is common Hermitian least squares solutions to matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$. From this results, we derived necessary and sufficient conditions for $X > (\geq, <, \leq)P$ in the LÖwner partial ordering. In particular, we obtained the maximal and minimal ranks and inertias of common Hermitian least squares solutions of (1.1) and identifying conditions for the existence of definite common Hermitian least squares solutions to (1.1).

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